Renormalization and resummation in the O(N) model

A. Jakovac [1]

Department of Physics, University of Wuppertal, D-42097 Wuppertal, Germany [2]

Abstract

In the O(N) model for the large N expansion one needs resummation which makes the renormalization of the model difficult. In the paper it is discussed, how can one perform a consistent perturbation theory at zero as well as at finite temperature with the help of momentum dependent renormalization schemes.

Key words: O(N) model, 2PI resummation, renormalization schemes
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In the O(N) model, using $1/N$ expansion we encounter non-convergent perturbative series. The source of the problem is that at one hand the coupling constant is proportional to the small parameter $1/N$, but, on the other hand the number of degrees of freedom is inversely proportional to it. As a net effect a contribution not suppressed by the small parameter can be formed on the radiative level. This spoils the strict perturbative ordering of loop levels, and so make the direct perturbation theory useless.

To the first level, the problem can be circumvent by introducing a new degree of freedom with the help of Hubbard Stratonovich transformation, and then determine the vacuum expectation value of this new field [1]. At higher order, however, this is not enough; what helps there is the modification of the propagation of the new degree of freedom. Then, however, special care is needed to maintain the renormalizability of the theory [2].

In this work we use the method developed in a series of papers to perform a renormalized 2PI resummation [3,4]. The idea is that we split the bare kernel into two parts, one is used as a kernel of the unperturbed theory, the other as a counterterm. Since by construction we do not change the bare theory, we keep the physics untouched. In [3] it is studied, how consistent is the perturbation theory with a generic momentum dependent counterterm. Shortly, the result is that if the unperturbed kernel can be power expanded

[1] e-mail:jakovacesr.phy.bme.hu
[2] On leave from Institute of Physics, BME Technical University, H-1521 Budapest, Hungary

around asymptotically large momenta, then we obtain a consistent renormalization procedure.

We start with the definition of the model: its Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \bar\Phi_i)(\partial^\mu \bar\Phi_i) - \frac{\bar m^2}{2}\bar\Phi_i\bar\Phi_i - \frac{\bar\lambda}{24N}\left(\bar\Phi_i\bar\Phi_i\right)^2. \tag{1}$$

We perform a Hubbard-Stratonovich transformation to introduce a new degree of freedom. For later convenience we perform an incomplete transformation; in imaginary time the resulting Lagrangian reads

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \bar\Phi_i)(\partial_\mu \bar\Phi_i) + \frac{\bar m^2}{2}\bar\Phi_i\bar\Phi_i + \frac{1}{2}\bar\chi^2 + \frac{i\bar g}{2\sqrt{N}}\bar\chi\bar\Phi_i\bar\Phi_i + \frac{\delta\lambda}{24N}\left(\bar\Phi_i\bar\Phi_i\right)^2, \tag{2}$$

where $\bar\lambda = 3\bar g^2 + \delta\lambda$. The $\delta\lambda$ term seemingly reintroduces the bad $1/N$ behavior, however we will allow its value to be at most $\mathcal{O}(1/N)$.

In the renormalization procedure must keep the bare Lagrangian intact. But we can redefine fields and render couplings to the renormalized and counterterm parts arbitrarily. For the field redefinition we use $\bar\Phi = Z^{1/2}\Phi$ and $\bar\chi = Z_\chi^{1/2}\chi_0 + iZ_\chi^{-1/2}\sqrt{N}\,q$, where $Z = 1+\delta Z$, $Z+\chi = 1+\delta Z_\chi$, and q is a c-number: it is necessary to cancel divergences proportional to the χ_0 field itself. We split the couplings as $Z\bar m^2 = m^2 + \delta m^2$ and $ZZ_\chi^{1/2}\bar g = g+\delta g$. To achieve resummation we split the quadratic χ_0 part in a momentum dependent way:

$$\frac{1}{2}Z_\chi|\chi_0(p)|^2 = \frac{1}{2}\chi_0^*(p)H(p)\chi_0(p) + \frac{1}{2}\chi_0^*(p)\delta H(p)\chi_0(p) \tag{3}$$

where the first term is considered to be part of the free Lagrangian, the second one is a momentum dependent counterterm. Finally we arrive at the form

$$\begin{aligned}\mathcal{L}_E =& \frac{1}{2}\Phi_i\left(-\partial^2 + m^2\right)\Phi_i + \frac{1}{2}\chi_0\, H(i\partial)\,\chi_0 + \frac{ig}{2\sqrt{N}}\chi_0\Phi_i\Phi_i + \sqrt{N}\, iq\chi_0 + \\ &+\frac{1}{2}\Phi_i\left(-\delta Z\partial^2 + \delta m^2\right)\Phi_i + \frac{1}{2}\chi_0\,\delta H(i\partial)\,\chi_0 + \frac{i\delta g}{2\sqrt{N}}\chi_0\Phi_i\Phi_i + \frac{\delta\lambda}{24N}(\Phi_i\Phi_i)^2. \end{aligned} \tag{4}$$

We assume that we are in the symmetry broken phase, where $m^2 < 0$.

As next we introduce nontrivial background for both Φ and χ. by rotating the coordinate system in the internal space we can achieve that

$$\chi_0 = -i\sqrt{N}\,X + \chi, \qquad \Phi_N = \sqrt{N}\Phi + \varrho, \qquad \Phi_i = \varphi_i\ (i = 1\dots N-1). \tag{5}$$

When we perform a perturbative analysis with the above Lagrangian, we find that the χ self-energy gets $\mathcal{O}(N^0)$ correction. In formula we obtain expansion

$$\Sigma_{\chi\chi}(p,\mathcal{E}) = -\frac{g^2}{2}I(p,\mathcal{E}) - \delta H_0(p), \tag{6}$$

where I is the finite temperature bubble diagram

$$I(p,\mathcal{E}) = \int\limits_q G_\pi(p-q)G_\pi(q), \qquad G_\pi(p) = \frac{1}{p^2+m_\pi^2}, \tag{7}$$

where $m_\pi^2 = m^2 + g^2 X^2$, and the integral is understood as $T\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{p}}{(2\pi)^3}$. The reason for the appearance of tree-level order correction is that although the coupling constant yields an $1/N$ factor, the number of participants in the loop (the φ_i fields) have a number of N, and the two factors cancel each other. A technically radiative, but in reality tree-level correction is a disaster for the perturbation theory, since this term can appear in any number as a subdiagram in any larger diagrams, and so the number of diagrams is infinite at each order. The common wisdom is that we have to perform a resummation in this case.

Formula (6) suggests a solution for this problem in the present, strictly perturbative framework. If we choose a specific scheme where

$$\delta H_0(p) = -\frac{g^2}{2} I(p,\mathcal{E}), \tag{8}$$

then the χ self-energy will be zero. A concrete calculation shows that $\delta H_0(p)$ asymptotically is $\sim \ln p$. According to (3) the χ-kernel $H(p)$ has the same momentum dependence as the counterterm, and so the consistency relation of [3] fulfills. The divergent part of $\delta H_0(p)$ determines, through (3), the divergent part of δZ_χ. The finite parts must be determined using the renormalization conditions, as usual.

One can also compute the free energy at the leading, $\mathcal{O}(N^0)$ order, and determine the necessary counterterms, q_0 and the free energy zero-point renormalization δf_0 (for details cf. [4]). The leading order counterterms then read

$$\begin{aligned}
\delta Z_{\chi,0} &= -\frac{g^2}{32\pi^2}\ln\frac{e\Lambda^2}{m_\varrho^2}, \qquad q_0 = \frac{g}{32\pi^2}\left[-\Lambda^2 + m^2\ln\frac{\Lambda^2}{m_\pi^2} + gX_{\min}\ln\frac{m_\varrho^2}{em_\pi^2}\right],\\
\delta f_0 &= -\frac{1}{32\pi^2}\left[m^2\Lambda^2 - \frac{m^4}{2}\ln\frac{\Lambda^2}{m_\pi^2} + \frac{g^2X_{\min}^2}{2}\ln\frac{m_\varrho^2}{em_\pi^2} - \frac{m_\pi^4}{4}\right],
\end{aligned} \tag{9}$$

where $X_{\min} = -m^2/g$ is the tree level minimum of the potential, $m_\varrho^2 = -2m^2$, and m_π^2 must be taken at $X_{\min}$. We can see that all the counterterm are local and independent of the temperature.

After this procedure the χ self-energy is zero at the $\mathcal{O}(N^0)$ order, and so the normal perturbative behavior is restored. Two peculiar remnants of the resummation problems, however, still survive. One is that the number of φ_i fields may still lift certain contributions to higher level of perturbation theory; but these never reach the tree level, and so there are just a finite number of diagrams affected in this way, which does not spoil the perturbative technique. The other peculiarity is that, as a consequence of the nontrivial χ propagator, the divergent structures of the counterterms may strongly deviate from the forms we are used to in the normal perturbative cases.

In the next-to-leading order we obtain the following self-energy for the φ_i fields:

$$\Sigma_\pi(p) = \frac{g^2}{N}\int\limits_q G_\pi(p-q)G_{\chi\chi}(q) + \delta Z_1 p^2 + \delta m_1^2 + \delta g_1 X + \frac{\delta\lambda_1}{6}\Phi^2 + \frac{\delta\lambda_1}{6}\int\limits_q G_\pi(q), \tag{10}$$

where $G_{\chi\chi}(p) = (p^2+m_\pi^2)/[\,H(p)(p^2+m_\pi^2)+g^2\Phi^2]$. Requiring finiteness we may determine the infinite parts of the counterterms in this formula. We obtain (cf. [4]):

$$\delta Z_1 = 0, \quad \delta\lambda_1 = 0, \quad \delta g_1 = \frac{2g}{N}\ln\ln\frac{L^2}{\Lambda^2},$$
$$\delta m^2_{1,div} = \frac{\Lambda^2}{N\ln L/\Lambda}\left[1 - \frac{1}{2\ln L/\Lambda} + \frac{1}{2\ln^2 L/\Lambda}\right] + \frac{2m^2}{N}\ln\ln\frac{L^2}{\Lambda^2}, \tag{11}$$

where $L = e^{\frac{16\pi^2}{g^2}} m_\varrho$ is the UV Landau pole position. Although these expressions are very different from the usual ones, these are local and independent of the temperature in this order, too.

The last term in (10) is formally a second-order contribution (one loop with a counterterm coupling), but again the number of degrees of freedom in the loop is N, which lifts up this contribution to the first order level. Direct analysis of this order using the original (non-resummed) degrees of freedom reveals the role of this term. The first term in (10), namely, represents an infinite number of diagrams, depicted on Fig. 1/a. Any

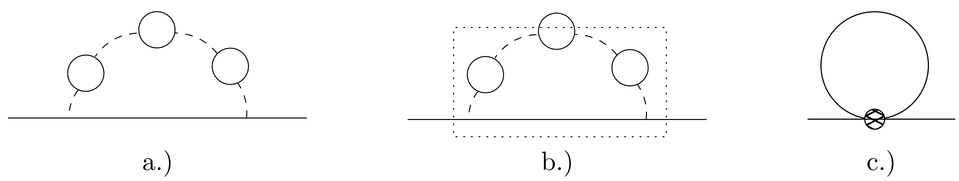

Fig. 1. The pion self-energy as expanded in the language of the non-resummed $\chi - \chi$ propagator (a.). Its potentially divergent subdiagram boxed by dotted line (b.) The diagram necessary to cancel this subdivergence (c.)

of these diagrams contains divergent subdiagrams, as shown by Fig. 1/b. The necessary counterterm is Fig. 1/c in all cases, which exactly corresponds to the last term of (10).

As a conclusion we may say that the use of momentum dependent perturbative schemes provide a compact way of handling resummations, with the benefit that it also provides a tool for renormalization. In case of O(N) model with $1/N$ expansion resummation is needed only at the leading order level, and it requires to determine the background field expectation values as well as the correct propagator for the χ field. At higher orders usual perturbative techniques can be used, the only remnant of a previous resummation are the appearance of certain diagrams at lower order level and the unusual divergence structure of the counterterms.

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